

SYMMETRIC ORTHOGONALITY AND CONTRACTIVE PROJECTIONS IN METRIC SPACES

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ABSTRACT. In this paper known result of symmetric orthogonality, as introduced by G. Birkhoff, and contractive nearest point projections from the linear are extended to the metric setting. If the space has non-positive curvature in the sense Pedersen or Busemann then it is shown that those concepts are actually equivalent. In the end using a recent characterization of Busemann convex spaces with weak angles in Foertsch et al (IMRN 2007) it is shown that the set of uniformly convex Busemann convex spaces with stable contractive projection property is exactly the set of $CAT(0)$ -spaces.

Orthogonality in the Euclidean setting can be described either by an angle condition or using Pythagoras theorem by a nearest point projection. More precisely, one says a geodesic γ intersects a geodesic η orthogonally if the intersection point is the closest point on η of all points γ_t . This condition appeared the first time for normed spaces in Birkhoff's work [Bir35]. He showed that orthogonality in terms of projections is, in general, not symmetric. Later the symmetric orthogonality was rediscovered in the setting of Hilbert geometries in [KP52] where it was called symmetric perpendicularity. A more recent reappearance was in the metric setting under the name of property (B) in [Kuw13].

An essential ingredient to obtain the contractive behavior for gradient flows of convex functionals for non-positively curved spaces is the fact that projections onto convex sets are themselves contractive (see bibliographic remarks of [Bač14, Chapter 5]). This result is well-known for Hilbert spaces. Kakutani showed [Kak39] (see also [Phe57]) that Hilbert spaces are the only (higher dimensional) Banach spaces having contractive projections onto convex sets. This property was subsequently generalized first to Hadamard manifolds and then to general $CAT(0)$ -spaces (see bibliographic remarks of [Bač14, Chapter 2]).

The contractive projection property can be also used to prove generalized “maximum principles” of harmonic maps into spaces with such a property. More precisely, if $h : M \rightarrow N$ is harmonic (in terms of minimal energy of the Dirichlet energy E) and $h(\partial M)$ is contained in a convex subset $C \subset N$ then $h(M)$ is contained in C . This follows from the contractive projection property and the characterization of the Dirichlet energy E as a Mosco limit of functionals E_ϵ satisfying $E_\epsilon(\pi_C \circ f) \leq E_\epsilon(f)$ for all maps f in the domain of E_ϵ where π_C is the nearest point projection onto C (see [Jos94, Chapter 8]).

After introducing the main concepts needed in this work, we show that the only higher dimensional Finsler manifolds with symmetric orthogonality are Riemannian manifolds (Proposition 4). Even stronger, the stable symmetric orthogonality

property requires the manifolds to be Riemannian regardless of their dimension (Corollary 7).

Afterwards we show that a space with contractive projections has always symmetric orthogonalities (Proposition 14). As it turns out these two properties are in fact equivalent if the spaces has non-positive curvature in the sense of Pedersen or Busemann (Theorem 18).

In the end, guided by ideas of [FLS07], we prove that the class of uniformly convex Busemann convex spaces with stable symmetric orthogonality (equivalently stable contractive projections) consists only of $CAT(0)$ -spaces (Theorem 22). Note that this result can be regarded as a non-smooth generalization of Proposition 4. Indeed, smooth $CAT(0)$ -spaces are Riemannian manifolds, so that $CAT(0)$ -spaces are Riemannian-like metric spaces. The main result of the paper can thus be summarized as saying that the stable symmetric orthogonality is a Riemannian property.

Preliminaries. Let (M, d) be a geodesic metric space, i.e. for each $x, y \in M$ there is a 1-Lipschitz map $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) = x$, $\gamma(1) = y$ and

$$d(\gamma_t, \gamma_s) = |s - t|d(x, y).$$

We say that γ is a $[0, 1]$ -*parametrized geodesic*. (M, d) is said to be *uniquely geodesic* if for each $x, y \in M$ there is exactly one geodesic connecting those points.

In the following we always assume that metric spaces are complete and geodesic and, if not mentioned otherwise, we implicitly assume geodesics are $[0, 1]$ -parametrized.

A subset $C \subset M$ is *weakly convex* if for all $x, y \in C$ there is a geodesic γ connecting x and y such that $\gamma_t \in C$. If all geodesics connecting x and y lie entirely in C then C is said to be *convex*. Note that the image of any geodesic is itself weakly convex.

Given a subset $C \subset M$ and $x \in M$, denote by $\pi_C(x)$ the set of closest points of x onto C , i.e.

$$\pi_C(x) = \{y \in C \mid d(x, y) = d(x, C) = \inf_{z \in C} d(x, z)\}.$$

If C is compact then $\pi_C(x)$ is always non-empty. If π_C is single-valued then we regard π_C as a (partially-defined) map. In general $\pi_C(x)$ is neither non-empty nor single-valued. A sufficient condition for π_C to be non-empty and single-valued for all closed convex sets is uniform ∞ -convexity.

We say (M, d) is *uniformly ∞ -convex* if there is a (strictly decreasing) function $\rho : (0, \infty) \rightarrow (0, \infty)$ such that for all $\epsilon > 0$ and all points $x, y, z \in M$ with

$$d(y, z) > \epsilon \max\{d(x, y), d(x, z)\}$$

then

$$d(x, m) \leq (1 - \rho(\epsilon)) \max\{d(x, y), d(x, z)\}$$

for all midpoints m of y and z

This definition is equivalent to uniform convexity in Banach spaces and was studied for metric spaces in [Foe04, Kel14]. Among several uniform convexity assumptions this is one of the weakest one. We only need such a condition as it ensures unique geodesics of the tangent spaces in terms of ultralimit blow-ups.

Every Riemannian and Finsler manifold is locally uniform ∞ -convexity as it is equivalent to strict convexity of balls in the smooth (finite dimensional) setting

(see [She97, Theorem 5.2]). A stronger condition demands for convexity of r -neighborhood of convex sets, i.e. we say that the metric space is *Pedersen convex*, or has *non-positive curvature in the sense of Pedersen* if for all (weakly) convex sets C , the sets

$$\bar{B}_r(C) = \{x \in M \mid d(x, C) \leq r\}, r \geq 0$$

are convex. Note that every weakly convex set is automatically convex. In particular, any geodesic γ is convex so that Pedersen convex metric spaces are uniquely geodesic. The condition was introduced in Pedersen's thesis [Ped52], see also [Bus55, Section 36] where it is called "a space with convex capsules".

A more popular and even stronger condition is called *Busemann convexity*, also called *non-positive curvature in the sense of Busemann* [Bus55, Section 36]. For this one require that for any geodesics γ and η the map

$$t \mapsto d(\gamma_t, \eta_t)$$

is convex. It is not difficult to see that Busemann convex spaces are always Pedersen convex the converse is not true. Examples are given by non-Riemannian Hilbert geometries (see [KS58]).

Busemann convexity gives already strong results, e.g. if $t \mapsto d(\gamma_t, \eta_t)$ is affine then the convex hull of γ and η is isometric to a convex subset of \mathbb{R}^2 equipped with a strictly convex norm (see [Bus55, Theorem (36.9)]). However, the condition is not stable under taking limit and it is not known if the tangent spaces of Busemann convex spaces are themselves Busemann convex.

A condition which is stable under limit operations is the *CAT(0)*-condition which can be formulated via comparison triangles (see [BH99, Chapter II.1]). More precisely, (M, d) is said to be a *CAT(0)*-space if for all $x, y, z \in M$ and $\tilde{x}, \tilde{y}, \tilde{z} \in \mathbb{R}^2$ satisfying

$$d(x, y) = \|\tilde{x} - \tilde{y}\|, d(x, z) = \|\tilde{x} - \tilde{z}\|, d(y, z) = \|\tilde{y} - \tilde{z}\|$$

it holds

$$d(x, m) \leq \|\tilde{x} - \tilde{m}\|$$

where m and \tilde{m} are the midpoints of y and z , resp. \tilde{y} and \tilde{z} .

Let ω be a *non-principle ultrafilter* on \mathbb{N} . We will regard ω as a finitely additive measure on \mathbb{N} such that for all $A \subset \mathbb{N}$ it holds $\omega(A) \in \{0, 1\}$ and $\omega(A) = 0$ whenever A is finite. Given a sequence $(a_n)_{n \in \mathbb{N}}$ in $\mathbb{R} \cup \{\pm\infty\}$. The ultrafilter ω "selects" exactly one converging subsequence of $(a_n)_{n \in \mathbb{N}}$. We denote the (unique) limit by $\lim_{\omega} (a_n)_{n \in \mathbb{N}}$ and called the *ultralimit* of (a_n) (w.r.t. ω).

With the help of ultralimits it is possible to define blow-up tangent spaces. More precisely, let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers with $\lambda_n \rightarrow 0$. Then the tangent space $(T_x^{(o)} M, d_x, x)$ at x (w.r.t. ultrafilter ω and the scale $(o) = (\lambda_n)_{n \in \mathbb{N}}$) is the ultralimit of the sequence of (pointed) metric spaces $(M, \lambda_n^{-1} d, x)_{n \in \mathbb{N}}$ where the points of $(T_x^{(o)} M, d_x, x)$ are just the set of sequences (x_n) with $\lim_{\omega} \lambda_n^{-1} d((x_n), x) < \infty$ and the metric is given by $d_x((x_n), (y_n)) = \lim_{\omega} \lambda_n^{-1} d(x_n, y_n)$. More precisely, the ultralimit is a set of equivalence classes since we identify two sequences (x_n) and (y_n) whenever $\lim_{\omega} \lambda_n^{-1} d(x_n, y_n) = 0$.

One can show that $(T_x^{(o)} M, d_x)$ is complete and it is geodesic whenever (M, d) is geodesic. Indeed, if (x_n) and (y_n) are points in $T_x^{(o)} M$ and (γ_n) a sequence of

geodesics connecting x_n and y_n then $(\gamma_n(t))$ is in $T_x^{(o)}M$ and

$$\lim_{\omega} \frac{1}{t} d((x_n), (\gamma_n(t))) = \frac{1}{1-t} \lim_{\omega} d((x_n), (\gamma_n(1-t))) = \lim_{\omega} d((x_n), (y_n))$$

so that $t \mapsto (\gamma_n(t))$ is a geodesic in $(T_x^{(o)}M, d_x)$.

SYMMETRIC ORTHOGONALITY

Definition 1 (Birkhoff orthogonal). A geodesic γ is said to be orthogonal to a geodesic η if they intersect in a common point $\gamma_0 = \eta_0$ and

$$\forall t, s \in [0, 1] : d(\eta_0, \gamma_t) \leq d(\eta_s, \gamma_t).$$

In other words, for all $t \in [0, 1]$ the point η_0 is the/a closest point of γ_t on η . In this case we write $\gamma \perp_{\gamma_0} \eta$.

For Riemannian manifolds it is known that if γ_0 is in the interior of γ and η , and γ and η stay sufficiently close to γ_0 then $\gamma \perp_{\gamma_0} \eta$ is equivalent to

$$g_{\gamma_0}(\dot{\gamma}_{\gamma_0}, \dot{\eta}_{\gamma_0}) = 0.$$

Therefore, $\eta \perp_{\gamma_0} \gamma$.

If the Riemannian manifold is a Hadamard spaces, i.e. it is simply connected and has non-positive sectional curvature, then $\gamma \perp_{p_0} \eta$ is equivalent to $g_{\gamma_0}(\dot{\gamma}_{\gamma_0}, \dot{\eta}_{\gamma_0}) = 0$ for all geodesics γ, η . However, the symmetry “ $\gamma \perp_{\gamma_0} \eta$ iff $\eta \perp_{\gamma_0} \gamma$ ” holds for general simply connected manifold without focal points. We leave the details to the interested reader.

Definition 2 (Symmetric Orthogonality). A geodesic space (M, d) is said to satisfy the symmetric orthogonality property (SO) if $\gamma \perp_{\gamma_0} \eta$ implies $\eta \perp_{\gamma_0} \gamma$. We say that $(SO)_{loc}$ holds if each point admits a neighborhood U such that the symmetry holds for all geodesics lying in U .

Remark. It suffices to assume that this property holds for geodesics starting at a fixed p_0 .

For any normed spaces satisfies (SO) if it satisfies $(SO)_{loc}$. Furthermore, for (reflexive) normed spaces whose dual space have strictly convex norm, property (SO) is equivalent to “ $\ell_v(w) = 0$ iff $\ell_w(v) = 0$ ” where ℓ_v and ℓ_w are the duals of v and resp. w obtained by

$$\begin{aligned} \ell_v(w') &= \lim_{\epsilon \rightarrow 0} \frac{\|v + \epsilon w'\|^2 - \|v\|^2}{2\epsilon} \\ \ell_w(v') &= \lim_{\epsilon \rightarrow 0} \frac{\|w + \epsilon v'\|^2 - \|w\|^2}{2\epsilon}. \end{aligned}$$

In Finsler geometry ℓ_v is equal to $g_v(v, \cdot)$ where g_v is called the fundamental tensor at v . We refer to [She97, Oht08] for all concepts of Finsler manifolds needed for the discussion below.

Lemma 3 (Linear Orthogonal Rigidity [Bir35, Jam47]). *If $(\mathbb{R}^n, \|\cdot\|)$ is a normed vector space with $n > 2$ then $(\mathbb{R}^n, \|\cdot\|)$ satisfies (SO) iff $(\mathbb{R}^n, \|\cdot\|)$ is Euclidean, i.e. the norm $\|\cdot\|$ is induced by an inner product.*

Actually this holds more general for any Finsler manifolds.

Proposition 4. *Assume (M, F) is a (smooth) Finsler manifold. Then (M, F) satisfies satisfying $(SO)_{loc}$ if and only if each tangent space $(T_x M, F_x)$ satisfies (SO) . In particular, (M, F) is either 2-dimensional or (M, F) is a Riemannian manifold.*

In case (M, F) is has non-positive flag curvature then the local condition $(SO)_{loc}$ implies global condition (SO) .

Remark. The proposition also holds for L^2 -products of Finsler manifolds which, in general, do not have C^2 -Finsler structure. However, the first variation and the local strong convexity of the square of the distance still holds for the factors and hence their product.

Proof. For simplicity assume F is symmetric. The proof relies only on the first variation formula and local strong convexity of the square of the distance and can be easily adapted to the asymmetric case.

Assume for some $x_0 \in M$ the tangent space $(T_{x_0} M, F_{x_0})$ does not satisfy property (SO) . Denote the fundamental tensor at $v \in TM$ by g_v . Since property (SO) does not hold, there are $v, w \in T_{x_0} M$ such that

$$g_v(v, w) = 0 \neq g_w(w, v).$$

Let $\gamma_v, \gamma_w : (-\epsilon, \epsilon)$ be geodesics with $\dot{\gamma}_v(0) = v$ and $\dot{\gamma}_w(0) = w$. By first variation we have

$$\begin{aligned} \frac{d}{d\epsilon} d^2(\gamma_v(-t), \gamma_w(\epsilon)) &= g_v(v, w) = 0 \\ \frac{d}{d\epsilon} d^2(\gamma_v(\epsilon), \gamma_w(-s)) &= g_w(w, v) \neq 0. \end{aligned}$$

By [She97, Theorem 5.2] (see also [Oht08, Corollary 5.2]) in a neighborhood U of x the square of the distance from a fixed point $x' \in U$ is strongly convex (in U) hence any critical point along a geodesic in U is automatically a global minimum (in U). Thus we see that $\gamma_v \perp_{\gamma_v(0)} \gamma_w$ but $\gamma_w \not\perp_{\gamma_v(0)} \gamma_v$.

Note that this also implies the converse as at least locally at x_0 the conditions $\gamma_v \perp_{x_0} \gamma_w$ and $g_v(v, w) = 0$ are equivalent.

In case (M, F) has non-positive flag curvature, the square of the distance from fixed points is strictly convex [Egl97] (see also remark after [Oht08, Corollary 5.2]). Therefore, $g_v(v, w) = 0$ iff $\gamma_v \perp_{x_0} \gamma_w$. \square

A well-known class of n -dimensional simply connected Finsler manifolds with non-positive flag curvature are Hilbert geometries (see [PT14] for an introduction to Hilbert geometries). One may verify that Hilbert geometries whose tangent norms satisfy everywhere (SO) must already be Riemannian manifolds and therefore isometric to the hyperbolic space. Using a very elegant short prove this fact was directly obtained by Kelly and Paige in [KP52].

Proposition 5 (Hyperbolic Orthogonal Rigidity in Hilbert Geometry [KP52]). *Any n -dimensional Hilbert geometry satisfying (SO) is isometry the n -dimensional hyperbolic space.*

The general condition (SO) does not exclude all non-Riemannian geometry. Nevertheless, it indicates that it is not stable, i.e. the L^2 -product of two spaces satisfying (SO) does in general not satisfy (SO) , not even locally. For that reason we define a stronger condition.

Definition 6 (Stable Symmetric Orthogonality). A geodesic space (M, d) is said to satisfy property (SO^*) the metric space $M \times_2 \mathbb{R} = (M \times \mathbb{R}, \tilde{d})$ satisfies property (SO) where $\tilde{d}((x, t), (y, s))^2 = d(x, y)^2 + |t - s|^2$. One may say that (M, d) satisfies (SO^*) if it satisfies the stable (SO) property. The local version will be denoted by $(SO^*)_{loc}$.

Corollary 7. *Every normed vector space and every Finsler manifold satisfying $(SO^*)_{loc}$ is a Riemannian manifold.*

More generally, we can show that the stable symmetric orthogonality (SO^*) implies the existence of symmetric angles if one-sided angles are well-defined. This is the case if the metric is strictly p -convex, $p > 1$, i.e. if for any $x \in M$ the map $y \mapsto d^p(x, y)$ is strictly convex.

Proposition 8. *Assume (M, d) is strictly p -convex for some $p > 1$. Then (M, d) satisfies (SO^*) iff for all unit speed geodesics $\gamma, \eta : [0, 1] \rightarrow M$ starting at $\gamma_0 = \eta_0$ it holds*

$$\lim_{t \rightarrow 0^+} \frac{d^2(\eta_t, \gamma_1) - d^2(\eta_0, \gamma_1)}{t} = \lim_{s \rightarrow 0^+} \frac{d^2(\eta_1, \gamma_s) - d^2(\eta_1, \gamma_0)}{s}.$$

Remark. This kind of commutativity condition was introduced in [OP14] and has strong implications on the behavior of gradient flows of convex functional. In particular, it shows that spaces satisfying (SO^*) must be Riemannian-like. Also note that the commutativity is stable under taking L^2 -product justifying the terminology “stable symmetric orthogonality”.

Proof. Note that p -convexity implies that $\partial_s^+ d^2(\eta_1, \gamma_s)|_{s=0}$ and $\partial_t^+ d^2(\eta_t, \gamma_1)|_{t=0}$ exists.

Denote the L^2 -product metric on $M \times \mathbb{R}$ also by d . Then there is an $a \in \mathbb{R}$ such that

$$\partial_s^+ d^2((\eta_1, 1), (\gamma_s, a \cdot s))|_{s=0} = 0.$$

Since $M \times_2 \mathbb{R}$ is also strictly p -convex [Foe04], we see that the closest point of $(\eta_1, 1)$ onto $s \mapsto (\gamma_s, a \cdot s)$ is $(\eta_0, 0) = (\gamma_0, 0)$.

Since orthogonality in $M \times_2 \mathbb{R}$ is symmetric we must have $\partial_s^+ d^2((\eta_t, t), (\gamma_1, a))|_{s=0} \geq 0$ implying

$$\partial_s^+ d^2(\eta_1, \gamma_s)|_{s=0} \leq \partial_t^+ d^2(\eta_t, \gamma_1)|_{t=0}.$$

Exchanging the roles of γ and η we obtain

$$\lim_{t \rightarrow 0^+} \frac{d^2(\eta_t, \gamma_1) - d^2(\eta_0, \gamma_1)}{t} = \lim_{s \rightarrow 0^+} \frac{d^2(\eta_1, \gamma_s) - d^2(\eta_1, \gamma_0)}{s}.$$

Conversely, if the commutativity conditions holds then it also holds for $M \times_2 \mathbb{R}$. Together with strict p -convexity one sees that (SO) holds for $M \times_2 \mathbb{R}$. Hence M satisfies (SO^*) . \square

Remark (Jensen’s inequality). In [Kuw13] Kuwae proved Jensen’s inequality for spaces satisfying (SO) . However, it seems that the proof of [Kuw13, Theorem 4.1] requires the stronger condition (SO^*) . Indeed it seems that [Kuw13, Lemma 2.12] cannot hold in general, because if an L^2 -product of two non-trivial spaces satisfies (SO) then both factors have to satisfy property $(SO^*)_{loc}$. Furthermore, a general L^p -product of smooth spaces with property (SO) can have at most dimension two. Therefore, for general $p \neq 2$, Jensen’s inequality on higher dimensional spaces seems still open.

In the following we focus only on the global version of (SO) . If (M, d) is locally convex then almost all results below hold w.r.t. their local version. However, assuming local convexity of (M, d) seems rather strong as there are spaces without convex set with interior, e.g. the Heisenberg group equipped with the Carnot-Caratheodory metric.

Another property appearing in [Kuw13, Definition 2.9] (in a slightly weaker form) is the following. In that paper property (SO) is called property (B) .

Definition 9 (Property (A)). A geodesic space (M, d) is said to satisfy (A) if for all weakly convex sets C it holds

$$d(x_C, y) \leq d(x, y)$$

for all $x \in M$, $y \in C$ and $x_C \in \pi_C(x)$. We say that (A^*) holds if $M \times_2 \mathbb{R}$ satisfies property (A) .

Lemma 10. *The conditions (A) and (SO) , resp. (A^*) and (SO^*) are equivalent.*

Proof. Assume first (SO) holds and let C be a weakly convex subset. Choose $x_C \in \pi_C(x)$ and $y \in C$. Let η be a geodesics connecting x_C and y in C and γ a geodesics γ connecting x_C and x . Since $\eta_t \in C$ and $x_C \in \pi_C(\gamma_s)$ we have so that

$$d(x_C, \gamma_s) = d(\eta_0, \gamma_s) \leq d(\eta_t, \gamma_s),$$

i.e. $\gamma \perp_{x_C} \eta$. Then (SO) implies $\eta \perp_{x_C} \gamma$ which is nothing but

$$d(x_C, y) = d(\gamma_0, y) \leq d(\gamma_1, y) = d(x, y).$$

Conversely, assume $\gamma \perp_{p_0} \eta$ for two geodesics starting at p_0 . Then let $x = \gamma_t$ and $y = \eta_1$ and note that $C = \cup_{t \in [0,1]} \{\eta_t\}$ is weakly convex and $p_0 \in \pi_C(x)$. If property (A) holds then

$$d(\eta_0, y) = d(p_0, y) \leq d(x, y) = d(\gamma_t, y)$$

and therefore $\eta \perp_{p_0} \gamma$. □

In the following we will directly use this equivalence without referring explicitly to this lemma.

CONTRACTIVE PROJECTIONS

Definition 11 (Contractive Projections). We say geodesic metric space (M, d) satisfies the contractive projection property (CP) if for all closed weakly convex sets C and all $x, y \in M$ it holds

$$d(x_C, y_C) \leq d(x, y)$$

whenever $x_C \in \pi_C(x)$ and $y_C \in \pi_C(y)$.

Remark. Property (CP) is well-known for linear spaces, see [Phe57] where Phelps showed that the class agrees with the class satisfying (SO) , compare also with Lemma 3. Furthermore, contractivity implies C must be (weakly) convex so that the assumption that C is (weakly) convex cannot be dropped.

Lemma 12. *If (M, d) satisfies (CP) then (M, d) is uniquely geodesic and the projection map π_C onto weakly convex sets C is at most single-valued. In particular, any weakly convex set is convex and π_C can be regarded as a contractive map whenever C is compact*

Proof. That π_C is at most single-valued follows from the definition. Furthermore, if there are unique geodesics between points then weak convexity implies convexity.

Assume $\gamma^{(1)}$ and $\gamma^{(2)}$ are two geodesics connecting two points x and y . Then the contraction property requires $\pi_{\gamma^{(1)}}(\gamma_t^{(2)}) = \gamma_t^{(1)}$ for $t \in [0, 1]$. If η is a geodesic connecting $\gamma_t^{(1)}$ and $\gamma_t^{(2)}$ then $\eta_s = \pi_{\gamma^{(1)}}(\eta_1)$. Because $x = \pi_{\gamma^{(1)}}(x)$, property (CP) implies

$$d(x, \eta_0) \leq d(x, \eta_t).$$

But then $\eta_0, \eta_1 \in \pi_\eta(x)$. Hence (CP) yields $\gamma_t^{(1)} = \gamma_t^{(2)}$. As $t \in [0, 1]$ is arbitrary, we see that there is exactly one geodesic connecting x and y . \square

Lemma 13. *If (M, d) satisfies (CP) then closed balls in (M, d) are strictly convex.*

Proof. Let $y \neq z \in \partial B_r(x)$ for some $x \in M$ and γ be a geodesic connecting y and z . By continuity of γ we see that $I = \gamma^{-1}(M \setminus \bar{B}_r(x))$ is open in $(0, 1)$.

We first show that $\bar{B}_r(x)$ is convex. For this it suffices to show that I is empty. Assume this is not the case. Then there a non-empty connected component of I given by $J = (a, b)$. The map given by $\tilde{\gamma} : t \mapsto \gamma((1-t)a + tb)$ is a geodesics connecting $\gamma(a)$ and $\gamma(b)$ with $\gamma(a) \neq \gamma(b) \in \partial B_r(x)$. Thus $\gamma(a), \gamma(b) \in \pi_{\tilde{\gamma}}(x)$ which contradicts uniqueness of $\pi_{\tilde{\gamma}}$.

It remains to show that $\gamma((0, 1)) \subset B_r(x)$. Note that the same argument as above shows that $\gamma \cap \partial B_r(x)$ cannot contain a non-trivial interval. In particular, there is a sequence $\epsilon_n \rightarrow 0$ such that $\gamma_{\epsilon_n}, \gamma_{1-\epsilon_n} \in B_{r_n}(x)$ for some $r_n < r$. By continuity we must have $r_n \rightarrow r$.

Now convexity of balls implies that $\gamma([\epsilon_n, 1 - \epsilon_n]) \subset \bar{B}_{r_n}(x) \subset B_r(x)$ so that

$$\gamma((0, 1)) = \bigcup_{n \in \mathbb{N}} \gamma([\epsilon_n, 1 - \epsilon_n]) \subset B_r(x).$$

\square

Proposition 14. *If (M, d) satisfies (CP) then it also satisfies (SO).*

Since smooth spaces are more rigid we obtain immediately the following corollary.

Corollary 15. *A Finsler manifold satisfying (CP) is either 2-dimensional or already Riemannian.*

Proof of the proposition. Assume (CP) and $\gamma \perp_{\gamma_0} \eta$, i.e. $\gamma_0 = \pi_\eta(\gamma_s)$, $s \in [0, 1]$. Since η is convex and $\eta_t = \pi_\eta(\eta_t)$, condition (CP) shows

$$d(\gamma_0, \eta_t) \leq d(\gamma_s, \eta_t).$$

However, this implies that $\gamma_0 = \pi_\gamma(\eta_t)$ and hence $\eta \perp_{\gamma_0} \gamma$. Because γ and η are arbitrary we see that (SO) holds. \square

The converse of the statement does not hold, not even if every ball is strictly convex. Indeed, if (M, d) is a closed ball of radius $R < \frac{\pi}{2}$ on the sphere \mathbb{S}^n with standard metric then its balls are strictly convex. However, the projections onto a geodesic is never contractive.

Nevertheless, assuming the space is Pedersen convexity we can show the converse.

Lemma 16. *Assume (M, d) is Pedersen convex and satisfies (SO). Whenever γ and η are two geodesics such that*

$$d(\gamma_t, \eta) = d(\eta_s, \gamma) \equiv \text{const}$$

then $d(\gamma_0, \gamma_1) = d(\eta_0, \eta_1)$.

Remark. In case the space is Busemann convex then the assumption (SO) can be dropped. Indeed, the assumptions would imply that γ and η span a flat rectangle, i.e. the convex hull of γ and η is isometric to a rectangle in a strictly convex normed space [Bus55, HL07] from which the claim immediately follows.

Proof. By symmetry of the assumption it suffices to show $d(\gamma_0, \gamma_1) \leq d(\eta_0, \eta_1)$. Let $t_0, t_1 \in [0, 1]$ such that $\eta_{t_i} \in \pi_\eta(\gamma_i)$, $i = 0, 1$. Possibly reversing η we can assume $t_0 \leq t_1$ and choose t_0 minimal and t_1 maximal in $[0, 1]$. Since $d(\eta_t, \gamma)$ is constant, one may verify using Pedersen convexity that $d(\gamma_t, \eta_{[t_0, t_1]}) = d(\gamma_t, \eta)$. In particular, we may restrict η to $[t_0, t_1]$ and can assume $\eta_i \in \pi_\eta(\gamma_i)$, $i = 0, 1$.

If $d(\gamma_0, \gamma_1) \leq d(\eta_0, \eta_1)$ then there is nothing to prove. By contradiction assume the converse. Let $\xi^{(i)}$ be the geodesics connecting γ_i and η_i . Then we have $\xi^{(i)} \perp_{\eta_i} \eta$. Since $c = d(\eta_i, \gamma) = d(\eta_i, \gamma_i)$ we also have $\xi^{(i)} \perp_{\eta_i} \gamma$. Thus by property (SO) also $\gamma \perp_{\eta_i} \xi^{(i)}$ and $\eta \perp_{\eta_i} \xi^{(i)}$.

Using continuity of the distance we can find an $s_0 \in (0, 1)$ such that

$$d(\eta_0, \eta_1) < d(\xi^{(0)}(s_0), \xi^{(1)}(s_0)) < d(\gamma_0, \gamma_1).$$

Denote the geodesic connecting $\xi^{(0)}(s_0)$ and $\xi^{(1)}(s_0)$ by ζ and note that Pedersen convexity implies that

$$\zeta_t \in \bar{B}_{s_0 c}(\gamma) \cap \bar{B}_{(1-s_0)c}(\eta).$$

In particular, $d(\zeta_t, \gamma) = d(\gamma_t, \zeta) \equiv s_0 c$ and $d(\zeta_t, \eta) = d(\eta_t, \zeta) \equiv (1 - s_0)c$. Thus again property (SO) shows $\xi^{(i)} \perp_{\xi^{(i)}(s_0)} \zeta$ and $\zeta \perp_{\xi^{(i)}(s_0)} \xi^{(i)}$.

Claim. For all $t \in [0, 1]$ it holds

$$\zeta_{dt}, \zeta_{1-d(1-t)} \in \pi_\zeta(\eta_t)$$

where $d = \frac{d(\eta_0, \eta_1)}{d(\zeta_0, \zeta_1)} < 1$. In particular, $I_t = \{s \in [0, 1] \mid \eta_s \in \pi_\zeta(\eta_t)\}$ contains an interval of width $(1 - d)$ and center $dt + \frac{1-d}{2}$.

Proof of the claim. Pedersen convexity implies that the sets $\bar{B}_r(\xi^{(i)})$, $i = 0, 1$, are convex for $r \geq 0$. Denote by π_r^i the projection onto $\bar{B}_r(\xi^{(i)})$. Since $\zeta \perp_{\eta_i} \xi^{(i)}$ we have

$$\begin{aligned} \zeta_{dt} &\in \pi_{td(\eta_0, \eta_1)}^0(\zeta_1) \\ \zeta_{1-d(1-t)} &\in \pi_{td(\eta_0, \eta_1)}^1(\zeta_0) \end{aligned}$$

where $i = 1 - j \in \{0, 1\}$. Thus if $\zeta_{t'} \in \pi_\gamma(\zeta_t)$ for $t' \geq dt$ then property (A) implies

$$(1 - s_0)c \leq d(\zeta_{dt}, \eta_t) \leq d(\zeta_{t'}, \eta_t) = (1 - s_0)c.$$

Hence $\gamma_{dt} \in \pi_\gamma(\eta_t)$ in this case. Similarly, if $t' \leq 1 - d(1 - t)$ then property (A) implies

$$(1 - s_0)c \leq d(\zeta_{1-d(1-t)}, \eta_t) \leq d(\zeta_{t'}, \eta_t) = (1 - s_0)c.$$

To conclude, just note that always $dt \leq 1 - d(1 - t)$ so that the opposite case above applies to $t' = 1 - d(1 - t)$ or resp. $t' = dt$. \square

The same argument also shows that

$$\gamma_{et}, \gamma_{1-e(1-t)} \in \pi_\gamma(\zeta_t)$$

where $e = \frac{d(\zeta_0, \zeta_1)}{d(\gamma_0, \gamma_1)} < 1$.

The claim can be used as follows: for sufficiently small $\epsilon > 0$ it holds

$$\begin{aligned}\zeta_{\frac{1}{2} \pm \epsilon} &\in \pi_{\zeta}(\eta_{\frac{1}{2}}) \\ \gamma_{\frac{1}{2}} &\in \pi_{\gamma}(\zeta_{\frac{1}{2} \pm \epsilon}).\end{aligned}$$

This, however, would imply that

$$d(\gamma_{\frac{1}{2}}, \zeta_{\frac{1}{2} \pm \epsilon}) + d(\zeta_{\frac{1}{2} \pm \epsilon}, \eta_{\frac{1}{2}}) = c = d(\gamma_{\frac{1}{2}}, \eta_{\frac{1}{2}}),$$

i.e. there is a geodesic connecting $\gamma_{\frac{1}{2}}$ and $\eta_{\frac{1}{2}}$ via $\zeta_{\frac{1}{2} \pm \epsilon}$. Since the points $\zeta_{\frac{1}{2} \pm \epsilon}$ are distinct we obtain two distinct geodesics connecting $\gamma_{\frac{1}{2}}$ and $\eta_{\frac{1}{2}}$ in contradiction to the unique geodesic character of Pedersen convex spaces. \square

Corollary 17. *A Pedersen convex metric spaces satisfying (SO) has strictly convex closed balls.*

Proof. Convexity of balls implies that whenever $t \mapsto d(x, \gamma_t)$, $t \in [0, 1]$ with $d(x, \gamma_0) = d(x, \gamma_1)$ attains its maximum in the interior of $[0, 1]$ then it is constant. To show strict convexity it suffices to show that whenever $t \mapsto d(x, \gamma_t)$ is constant then so is $t \mapsto \gamma_t$. Let $\eta : t \mapsto x$ the constant geodesic at x . Then the assumptions of the previous theorem hold for γ and η so that

$$d(\gamma_0, \gamma_1) = d(\eta_0, \eta_1) = 0.$$

Therefore, $\gamma \equiv \eta$ is a trivial geodesic. \square

Theorem 18. *A Pedersen convex metric space satisfies (SO) iff it satisfies (CP). In particular, this result holds for Busemann convex spaces.*

Remark. It is not clear whether Pedersen convexity of M implies Pedersen convexity of $M \times_2 \mathbb{R}$. Thus (SO^*) may not imply (CP^*) . Nevertheless, $M \times_2 \mathbb{R}$ is Busemann convex spaces whenever M is so that (SO^*) and (CP^*) are equivalent for Busemann convex spaces.

Proof. We only need to show that Pedersen convex spaces that satisfy (SO) also satisfy (CP).

Choose $x_0, y_0 \in M$ and let C be a closed convex sets such that $x_C \in \pi_C(x)$ and $y_C = \pi_C(y)$ are well-defined. Assume w.l.o.g. $m = d(x_0, C) \leq d(y_0, C)$.

Set $C_r = \bar{B}_r(C)$ and note that $\pi_{C_r}(x)$ and $\pi_{C_r}(y)$ are non-empty as they contain points on the geodesics connecting x and x_C and resp. y and y_C . If $d(x_0, C) < d(y_0, C)$ then for any $\tilde{y}_0 \in \pi_{C_m}(y)$ it holds $d(x_0, C) = d(\tilde{y}_0, C)$ and since $x_0 \in C_m$ property (A) yields

$$d(x_0, \tilde{y}_0) \leq d(x_0, y_0).$$

Thus it suffices to show that $d(x_C, y_C) \leq d(x_0, y_0)$ whenever $d(x_0, C) = d(y_0, C)$.

Let $t \mapsto x_t$ and $t \mapsto y_t$ be the geodesics connecting x_0 and x_C and resp. y_0 and y_C . Note that $x_t \in \pi_{C_{m(1-t)}}(x_0)$ and $y_t \in \pi_{C_{m(1-t)}}(y_0)$. We will show that $t \mapsto d(x_t, y_t)$ is decreasing.

Denote by $\gamma^{(t)}$ the geodesic connecting x_t and y_t for $t \in [0, 1]$. Pedersen convexity implies

$$d(\gamma^{(t)}(s), C) \leq d(x_t, C) = m(1-t) \quad \text{for all } s \in [0, 1].$$

If

$$d(\gamma^{(t)}(s), C) = m(1-t) \quad \text{for all } s \in [0, 1]$$

then Pedersen convexity shows that for all $\epsilon \in [0, 1 - t]$ it holds

$$d(\gamma^{(t+\epsilon)}(s), C) = m(1 - (t + \epsilon))$$

and

$$d(\gamma^{(t+\epsilon)}(s), \gamma^{(t)}) = d(\gamma^{(t)}(s), \gamma^{(t+\epsilon)}) = m\epsilon.$$

By the previous lemma we have $d(x_{t+\epsilon}, y_{t+\epsilon}) = d(x_t, y_t)$.

Assume there is an $s \in (0, 1)$ such that $d(\gamma^{(t)}(s), C) < m(1 - t)$. In particular, for all sufficiently small $\epsilon > 0$ there is an $s_\epsilon \in (0, 1)$ such that $d(\gamma^{(t)}(s_\epsilon), C) = m(1 - (t + \epsilon))$. Since $\gamma^{(t)}(s_\epsilon) \in C_{m(1-(t+\epsilon))}$ we obtain

$$\begin{aligned} d(x_{t+\epsilon}, \gamma^{(t)}(s_\epsilon)) &\leq d(x_t, \gamma^{(t)}(s_\epsilon)) \\ d(y_{t+\epsilon}, \gamma^{(t)}(s_\epsilon)) &\leq d(y_t, \gamma^{(t)}(s_\epsilon)) \end{aligned}$$

where $x_{s_i} \in \pi_{C_r}(x_{t_i})$. Since $\gamma^{(t)}(s_\epsilon)$ is the s_ϵ -midpoint of x_t and y_t , triangle inequality shows $d(x_{t+\epsilon}, y_{t+\epsilon}) \leq d(x_t, y_t)$.

In either case we have shown that $t \mapsto d(x_t, y_t)$ is decreasing. In particular, $d(x_C, y_C) \leq d(x_0, y_0)$. Since x and y are arbitrary we see that (CP) holds. \square

Since any Pedersen convex Riemannian manifold is a Hadamard manifold, i.e. a simply connected Riemannian manifold of non-positive curvature, we obtain in combination with Proposition 4.

Corollary 19. *The only Pedersen convex Finsler manifolds of dimension greater than two with contractive projections are Hadamard manifolds.*

The theorem shows that Pedersen convexity together with property (SO) implies that the gradient flow of $x \mapsto d(x, C)$ is contractive. In the smooth setting, Ohta-Sturm [OS12] showed that gradient flows of convex functions are, in general, not contractive for non-Riemannian Finsler manifolds. In the Riemannian-like setting with generalized Ricci curvature bounds, contractiveness of gradient flow $x \mapsto d(x, C)$ is equivalent to convexity [Stu14, Corollary 2]. Thus we could ask the following.

Conjecture 20. *Assume (M, d) is a (uniformly ∞ -convex) Pedersen convex space satisfying (SO^*) . Then the gradient flows of convex functions are contractive.*

Since any Pedersen convex Riemannian manifold is already Busemann convex we also conjecture.

Conjecture 21. *Assume (M, d) is Pedersen convex and satisfies (SO^*) . Then (M, d) is Busemann convex and thus satisfies (CP^*) .*

Together with the result of the next section one could show that uniformly ∞ -convex Pedersen convex spaces satisfying (SO^*) are $CAT(0)$ -spaces.

Other projection properties. In [KR13] the property called double projections property was investigated. This property roughly requires that for pair of points (q, \bar{q}) of two convex sets C_1 and C_2 the condition $d(q, \bar{q}) = d(C_1, C_2)$ is equivalent to requiring that (q, \bar{q}) satisfy $\pi_{C_1}(\bar{q}) = q$ and $\pi_{C_2}(q) = \bar{q}$. It was shown [KR13, Theorem 2.2] that this property holds for linear spaces equipped with a strictly convex C^2 -norm.

Actually it is possible to prove this for all (finite dimensional) Busemann convex Finsler manifolds. One only needs (quasi-)convexity of $t \mapsto d(\gamma_t, \eta_t)$ for geodesics

γ in C_1 starting at q and η in C_2 starting at \bar{q} . Indeed, the (quasi-)convexity property implies $\frac{d}{dt}d(q, \eta_t)|_{t=0} \geq 0$ and $\frac{d}{dt}d(\gamma_t, \bar{q})|_{t=0} \geq 0$. Thus $\frac{d}{dt}d(\gamma_t, \eta_t)|_{t=0} = \frac{d}{dt}d(\gamma_t, \bar{q})|_{t=0} + \frac{d}{dt}d(q, \eta_t)|_{t=0} \geq 0$ which shows that $d(q, \bar{q})$ has a global minimum at (q, \bar{q}) . Note that $t \mapsto d(\gamma_t, \eta_t)$ is differentiable since there are no conjugate/focal points in a Busemann convex Finsler manifold.

BUSEMANN CONVEX SPACES SATISFYING (CP^*)

In this section we are going to prove our main result.

Theorem 22. *Let (M, d) be a complete geodesic metric space which is uniformly ∞ -convex. Then the following are equivalent:*

- (1) (M, d) is a $CAT(0)$ -space
- (2) (M, d) is a Busemann convex space satisfying (CP^*)
- (3) (M, d) is a Busemann convex space satisfying (SO^*) .

We first prove the following lemma on stability of the condition (SO) .

Lemma 23. *Let $(M_n, d_n, x_n)_{n \in \mathbb{N}}$ be a sequence of geodesic metric spaces satisfying (SO) (resp. (SO^*)). If the ultralimit $\lim_\omega(M_n, d_n, x_n)$ is uniquely geodesic then it satisfies (SO) (resp. (SO^*)).*

Proof. The $*$ -version follows by noting that

$$\left(\lim_\omega(M_n, d_n, x_n) \times_2 (\mathbb{R}, |\cdot|, 0) \right) = \left(\lim_\omega(M_n, d_n, x_n) \right) \times_2 (\mathbb{R}, |\cdot|, 0).$$

Since $(M_\omega, d_\omega, x_\omega) = \lim_\omega(M_n, d_n, x_n)$ is uniquely geodesic, any geodesic in (M_ω, d_ω) is given by an ultralimit of a sequence geodesics γ_n . So it suffices to show for geodesic (γ_n) and (η_n) in M_ω with $(\gamma_n) \perp_{(p_n)} (\eta_n)$ also $(\eta_n) \perp_{(p_n)} (\gamma_n)$ holds.

Assume $(\gamma_n) \perp_{(p_n)} (\eta_n)$, i.e. for all $s, t \in [0, 1]$

$$\lim_\omega d_n(\eta_n(s), \gamma_n(t)) \geq \lim_\omega d_n(p_n, \gamma_n(t)).$$

Fix $t \in [0, 1]$, let q_n be a closest point of $\gamma_n(t)$ on η_n . Then $d_n(q_n, \gamma_n(t)) \leq d_n(p_n, \gamma_n(t))$ so that

$$\lim_\omega d_n(p_n, \gamma_n(t)) = \lim_\omega d_n(q_n, \gamma_n(t)).$$

Denote by $\tilde{\gamma}_n$ the geodesic connecting q_n and $\gamma_n(t)$. Then property (SO) for (M_n, d_n) implies $\eta_n \perp_{q_n} \tilde{\gamma}_n$. In particular, since $\tilde{\gamma}_n(1) = \gamma_n(t)$ it holds

$$d_n(\eta_n(s), \gamma_n(t)) \geq d_n(\eta_n(s), q_n) \quad \text{for all } s \in [0, 1].$$

Combining the above we obtain for the ultralimit

$$\lim_\omega d_n(\eta_n(s), \gamma_n(t)) \geq \lim_\omega d_n(\eta_n(s), q_n) = \lim_\omega d_n(\eta_n(s), p_n)$$

for all $s \in [0, 1]$. Since $t \in [0, 1]$ was arbitrary it follows that $(\eta_n) \perp_{p_n} (\gamma_n)$. Thus property (SO) holds for (M_ω, d_ω) . \square

Lemma 24. *Assume (M_n, d_n, x_n) is a sequence of uniformly ∞ -convex metric spaces with same uniformity function ρ then any ultralimit $\lim_\omega(M_n, d_n, x_n)$ is uniformly ∞ -convex with uniformity function ρ .*

Proof. Let $(x_n), (y_n)$ and (z_n) be three points in $(M_\omega, d_\omega, x_\omega) = \lim_\omega (M_n, d_n, x_n)$ with

$$d_\omega((y_n), (z_n)) > \epsilon \max\{d_\omega((x_n), (y_n)), d_\omega((x_n), (z_n))\}.$$

Assume w.l.o.g. $d_\omega((x_n), (y_n)) \geq d_\omega((x_n), (z_n))$. Then $\omega(A) = 1$ for

$$A = \{n \in \mathbb{N} \mid d_n(y_n, z_n) > \epsilon d_n(x_n, y_n)\}.$$

Now let (w_n) be a midpoint of (y_n) and (z_n) (w.r.t. d_ω). Note that w_n may not be a midpoint m_n of y_n and z_n .

We claim that $(w_n) = (m_n)$ in M_ω . Assume by contradiction this is not the case. Then $d_\omega((m_n), (w_n)) > 0$ and thus $d_\omega((y_n), (z_n)) > 0$. So for some $\delta > 0$ it holds $\omega(B) = 1$ where

$$\begin{aligned} B &= \{n \in A \mid d_n(m_n, w_n), d_n(y_n, z_n) \geq \delta, \\ &\quad d_n(y_n, w_n), d_n(z_n, w_n) \leq \frac{1}{2}(d_n(y_n, z_n) + \rho)\} \end{aligned}$$

with

$$\rho = \frac{\rho(\delta/2)\delta}{2(1 - \rho(\delta/2))} > 0.$$

Note that we used the fact that (w_n) is a midpoint of (y_n) and (z_n) .

Let (v_n) be the sequence of midpoints of m_n and w_n . Then by uniform convexity

$$\begin{aligned} d_n(y_n, v_n) &\leq (1 - \rho(\delta/2)) \max\{d_n(y_n, w_n), \frac{1}{2}d_n(y_n, z_n)\} \\ d_n(z_n, v_n) &\leq (1 - \rho(\delta/2)) \max\{d_n(z_n, w_n), \frac{1}{2}d_n(y_n, z_n)\} \end{aligned}$$

for $n \in B$. But then

$$\begin{aligned} d_n(y_n, v_n) + d_n(z_n, v_n) &\leq \frac{1}{2}(1 - \rho(\delta/2))(d_n(y_n, z_n) + \rho) \\ &\leq d_n(y_n, z_n) - \rho(\delta/2)(d_n(y_n, z_n) - \frac{\delta}{2}) \\ &< d_n(y_n, z_n) \end{aligned}$$

for $n \in B$ which contradicts the triangle inequality. Thus it holds $(m_n) = (w_n)$.

Then uniform convexity implies

$$d_n(x_n, m_n) \geq (1 - \rho(\epsilon))d_n(x_n, y_n)$$

for $n \in B$ so that

$$d_\omega((x_n), (w_n)) \geq (1 - \rho(\epsilon))d_\omega((x_n), (y_n)).$$

□

Proof of Theorem 22. The following proof relies heavily on [FLS07, Section 5]. To keep the paper short we use those concepts without further introducing them.

Note that every $CAT(0)$ -space satisfies (CP^*) and (SO^*) and by the previous section the last two statements of the theorem are equivalent for Busemann convex spaces.

So assume (M, d) is a Busemann convex metric space satisfying (SO^*) . Fix an ultrafilter ω and a scale $(\lambda_n)_{n \in \mathbb{N}}$ with $\lambda_n \rightarrow 0$. By the two lemmas above the tangent spaces $(T_x^{(o)} M, d_x, x)$ are also uniformly ∞ -convex and satisfy (SO^*) . Since the ultralimit of Busemann convex spaces is often convex ([Kle99, Definition 10.5],

see also [FLS07, (OC1-3)]), they are also Busemann convex as uniformly ∞ -convex spaces have unique geodesics.

Thus any two (ultra)rays starting at x in $T_x^{(o)}M$ form a flat 2-dimensional sector C which is convex in $(T_x^{(o)}M, d_x)$. However, any such C must satisfy (SO^*) and is therefore isometric to a Euclidean sector. Thus weak angles exists between geodesics in (M, d) (see [FLS07, Lemma 5.1]) so that [FLS07, Proposition 5.2] implies that (M, d) must be a $CAT(0)$ -space. \square

Generalizations. The assumption of uniform ∞ -convexity can be dropped if it is possible to show the following.

Conjecture 25. *Assume (M, d) is Busemann convex and γ and η are two geodesics starting at x . Let $\bar{\gamma}$ and $\bar{\eta}$ the corresponding ultrarays in $T_x^{(o)}M$. Then there is a (weakly convex) 2-dimensional flat sector C containing $\bar{\gamma}$ and $\bar{\eta}$ such that each two points of C are connected by (straight line) geodesic given as ultralimits of geodesics in (M, d) .*

Indeed, $C \times_2 \mathbb{R}$ is often convex along the ultralimits of geodesics in $M \times_2 \mathbb{R}$ (see [Kle99, FLS07]) and the proof of Lemma 23 shows that (SO) holds for those geodesics. Since Lemma 3 (see [Jam47]) only needs (SO) for the straight lines it follows that $C \times_2 \mathbb{R}$ is Euclidean. Then again [FLS07, 5.1, 5.2] gives the result. Since (SO) does not imply unique geodesics between points, it is unclear whether Busemann convexity could be replaced with the requirement that the space is often convex in order to show that (SO^*) implies (CP^*) .

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